Refined singular moduli through p-adic deformations

Michael Daas

Université du Luxembourg
Max-Planck-Institut für Mathematik

August 3rd, 2026

Rational singular moduli

CM-theory

Let $K/\mathbb{Q}$ be an imaginary quadratic field and let E be an elliptic curve with CM by $\mathcal{O}_K \subset K$. Then $K(j(E)) = H_K$, the Hilbert class field of K.

Discriminant $-D$ of K	Conductor f of R	j-invariant of E
-3	1	0
	2	$2^4 3^3 5^3$
	3	$-2^{15} 3 \cdot 5^3$
-4	1	$2^6 3^3$
	2	$2^3 3^3 11^3$
-7	1	$-3^3 5^3$
	2	$3^3 5^3 17^3$
-8	1	$2^6 5^3$
-11	1	-2^{15}
-19	1	$-2^{15} 3^3$
-43	1	$-2^{18} 3^3 5^3$
-67	1	$-2^{15} 3^3 5^3 11^3$
-163	1	$-2^{18} 3^3 5^3 23^3 29^3$

Differences between singular moduli (1/2)

Discriminant $-D$ of K	Conductor f of R	j-invariant of E
-7	1	$-3^3 5^3$
-11	1	-2^{15}
-19	1	$-2^{15} 3^3$
-43	1	$-2^{18} 3^3 5^3$

We compute differences between singular moduli:

$$
\begin{aligned}
j(E_7) - j(E_{11}) &= 7 \cdot 13 \cdot 17 \cdot 19; \\
j(E_7) - j(E_{19}) &= 3^7 \cdot 13 \cdot 31; \\
j(E_7) - j(E_{43}) &= 3^6 \cdot 5^3 \cdot 7 \cdot 19 \cdot 73; \\
j(E_{11}) - j(E_{19}) &= 2^{16} \cdot 13; \\
j(E_{11}) - j(E_{43}) &= 2^{15} \cdot 7^2 \cdot 19 \cdot 29; \\
j(E_{19}) - j(E_{43}) &= 2^{15} \cdot 3^6 \cdot 37.
\end{aligned}
$$

Differences between singular moduli (2/2)

Discriminant $-D$ of K	Conductor f of R	j-invariant of E
-19	1	$-2^{15}3^3$
-43	1	$-2^{18}3^3 5^3$
-67	1	$-2^{15}3^3 5^3 11^3$
-163	1	$-2^{18}3^3 5^3 23^3 29^3$

We compute more differences between singular moduli:

$$\begin{aligned}
j(E_{19}) - j(E_{43}) &= 2^{15} \cdot 3^6 \cdot 37;\\
j(E_{19}) - j(E_{67}) &= 2^{16} \cdot 3^7 \cdot 13 \cdot 79;\\
j(E_{19}) - j(E_{163}) &= 2^{15} \cdot 3^7 \cdot 13 \cdot 19 \cdot 31 \cdot 37 \cdot 67 \cdot 193;\\
j(E_{43}) - j(E_{67}) &= 2^{15} \cdot 3^6 \cdot 5^3 \cdot 7^2;\\
j(E_{43}) - j(E_{163}) &= 2^{19} \cdot 3^6 \cdot 5^3 \cdot 7^3 \cdot 37 \cdot 433;\\
j(E_{67}) - j(E_{163}) &= 2^{15} \cdot 3^7 \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 139 \cdot 331.
\end{aligned}$$

Gross–Zagier

Theorem (Gross–Zagier, 1984)

Let E_1 and E_2 be elliptic curves with CM by maximal orders of coprime discriminants D_1 and D_2 respectively with Hilbert class fields H_1 and H_2. Then

$$\mathrm{Nm}_{\mathbb{Q}}^{H_1H_2}(j(E_1) - j(E_2)) = \prod_{|x| < \sqrt{D}} \mathcal{F}\left(\frac{D - x^2}{4}\right),$$

where $D = D_1D_2$ and $\mathcal{F}(n)$ for $n \in \mathbb{N}$ is either the power of a prime $\ell \mid n$, or 1.

For $(D_1, D_2) = (-7, -11)$, we have $j(E_1) - j(E_2) = 7 \cdot 13 \cdot 17 \cdot 19$.

x	± 1	± 3	± 5	± 7
$(D - x^2)/4$	19	17	13	7

For $(D_1, D_2) = (-7, -19)$, we have $j(E_1) - j(E_2) = 3^7 \cdot 13 \cdot 31$.

x	± 1	± 3	± 5	± 7	± 9	± 11
$(D - x^2)/4$	$3 \cdot 11$	31	3^3	$3 \cdot 7$	13	3

Setting up

Let $D_1, D_2 < 0$ be coprime fundamental discriminants and write $D = D_1 D_2$.

$$K_1 = \mathbb{Q}(\sqrt{D_1}), \quad K_2 = \mathbb{Q}(\sqrt{D_2}),$$
$$F = \mathbb{Q}(\sqrt{D}), \quad L = \mathbb{Q}(\sqrt{D_1}, \sqrt{D_2}).$$

$$\begin{tikzcd}[arrows=dash]
 & L \arrow[dl] \arrow[d, "\chi"] \arrow[dr] & \\
K_1 \arrow[dr] & F \arrow[d] & K_2 \arrow[dl] \\
 & \mathbb{Q} &
\end{tikzcd}$$

Let χ be the genus character of L/F: if $\mathfrak{p} \subset \mathcal{O}_F$ is prime, then

$$\chi(\mathfrak{p}) = \begin{cases} 1 & \text{if } \mathfrak{p} \text{ splits in } L/F; \\ -1 & \text{if } \mathfrak{p} \text{ is inert in } L/F. \end{cases}$$

Let $\mathcal{D}_F$ denote the different ideal of F.

Zagier's proof

First step is to rewrite the theorem as

$$\log \mathrm{Nm}_{\mathbb{Q}}^{H_1 H_2}\big(j(E_1) - j(E_2)\big) = \sum_{\substack{\nu \in \mathcal{D}_F^{-1,+} \\ \mathrm{tr}(\nu)=1}} \sum_{I \mid (\nu)\mathcal{D}_F} \chi(I) \log \mathrm{Nm}(I).$$

This looks like the diagonal restriction of the weight k Hilbert Eisenstein series attached to the trivial character $\mathbb{1}$ and χ:

$$E_{\mathbb{1},\chi}(z,z,k) = \mathrm{const} + \sum_{n=1}^{\infty} \left(\sum_{\substack{\nu \in \mathcal{D}_F^{-1,+} \\ \mathrm{tr}(\nu)=n}} \sum_{I \mid (\nu)\mathcal{D}_F} \chi(I)\mathrm{Nm}(I)^{k-1} \right) q^n.$$

"One line proof:"

$$e^{\mathrm{hol}} \left(\frac{d}{ds} E_{\mathbb{1},\chi}(z,z,s)\Big|_{s=0} \right) \in M_2(\mathrm{SL}_2(\mathbb{Z})) = \{0\}.$$

Its Fourier coefficients involve two terms, one for each side $\implies$ equal.

Shimura curves

The j-function generates the function field of the modular curve

$$j : SL_2(\mathbb{Z}) \setminus \mathcal{H} = Y_1(\mathbb{C}) \xrightarrow{\sim} \mathbb{A}^1,$$

attached to the quaternion algebra $M_2(\mathbb{Q})$. Now let $\mathfrak{B}$ be an *indefinite* quaternion algebra with discriminant $N \neq 1$. Let $\mathfrak{R}_1$ the units of norm 1 in a maximal order. Then

$$X_N(\mathbb{C}) = \mathfrak{R}_1 \setminus \mathcal{H};$$

is an example of a *Shimura curve*, which is an algebraic curve $/\mathbb{Q}$.

Proposition

The Shimura curve X_N is of genus 0 if and only if $N \in \{6, 10, 22\}$.

Assume that $g(X_N) = 0$, so there exists

$$j_N : X_N \xrightarrow{\sim} \mathbb{P}^1.$$

To have CM points, need the two primes $p, q \mid N$ to be inert in both K_1 and K_2. Experiment for τ_1, τ_1' CM-points of disc. -43, and τ_2, τ_2' of disc. -163 on X_6:

$$\mathrm{Nm}_{\mathbb{Q}}^{F}\left[\frac{j_N(\tau_1) - j_N(\tau_2)}{j_N(\tau_1') - j_N(\tau_2)} \cdot \frac{j_N(\tau_1') - j_N(\tau_2')}{j_N(\tau_1) - j_N(\tau_2')}\right] = \left(\frac{2 \cdot 29 \cdot 257 \cdot 277}{73 \cdot 137 \cdot 241}\right)^2.$$

Čerednik–Drinfeld

Let B be a *definite* quaternion algebra with discriminant M and let $R \subset B$ be a maximal order. Let $p \nmid M$ be a prime and consider the group $\Gamma = R[1/p]_1$ of units of norm 1. Now form the quotient

$$\Gamma \setminus \mathcal{H}_p,$$

where $\mathcal{H}_p = \mathbb{P}^1(\mathbb{C}_p) \setminus \mathbb{P}^1(\mathbb{Q}_p)$ is the *p-adic upper half plane*.

Theorem (Čerednik–Drinfeld)

The quotient $\Gamma \setminus \mathcal{H}_p$ as a rigid p-adic space is isomorphic to $X_{Mp}(\mathbb{C}_p)$.

For $N = \{6, 10, 22\}$ and $p \mid N = pq$, this yields a purely p-adic viewpoint.

Question

Which functions on $\Gamma \setminus \mathcal{H}_p$ correspond to j_N on the other side?

Let $w_1, w_2 \in \mathcal{H}_p$. Then consider the expression

$$\Theta(w_1, w_2; z) = \prod_{\gamma \in \Gamma} \frac{z - \gamma w_1}{z - \gamma w_2}.$$

If $N \in \{6, 10, 22\}$, this defines a function on $\Gamma \setminus \mathcal{H}_p$ with divisor $[w_1] - [w_2]$.

The Giampietro–Darmon Conjecture

Purely by comparing divisors on genus 0 curves,

$$\Theta(w_1, w_2; z) = c(w_1, w_2) \cdot \frac{j_N(z) - j_N(w_1)}{j_N(z) - j_N(w_2)}, \text{ for some } c(w_1, w_2) \in \mathbb{C}_p.$$

Now choose $w_1 = \tau_1$ and $w_2 = \tau_1'$; its Galois conjugate. Because we don't know $c(\tau_1, \tau_1')$, we opt to study instead

$$\frac{j_N(\tau_2) - j_N(\tau_1)}{j_N(\tau_2) - j_N(\tau_1')} \frac{j_N(\tau_2') - j_N(\tau_1')}{j_N(\tau_2') - j_N(\tau_1)} = \prod_{\gamma \in \Gamma} \frac{\tau_2 - \gamma\tau_1}{\tau_2 - \gamma\tau_1'} \frac{\tau_2' - \gamma\tau_1'}{\tau_2' - \gamma\tau_1} =: \Theta(\tau_1, \tau_2).$$

One can p-adically approximate this and recognise it as an algebraic number.

Conjecture (Giampietro–Darmon, 2022)

It holds that $\Theta(\tau_1, \tau_2) \in H_1 H_2$ is of degree $2h_1h_2$ over $\mathbb{Q}$, and

$$\mathrm{Nm}_{\mathbb{Q}}^{H_1H_2}(\Theta(\tau_1, \tau_2)) = \prod_{|x| < \sqrt{D}} \mathcal{F}\left(\frac{D - x^2}{4N}\right)^{\delta(x)},$$

where $\delta(x) \in \{\pm 1\}$ is a sign determined by $x \mod 4N$.

Rigid meromorphic cocycles (1/2)

The CM-values of the j-function have two different properties:

- They generate the Hilbert class field: $H_K = K(j(E))$;
- Differences of two CM-values display interesting factorisations.

Cohomologically, we may write

$$j \in H^0(SL_2(\mathbb{Z}), \mathcal{M}^\times),$$

where $\mathcal{M}^\times$ is the group of non-zero meromorphic functions on $\mathcal{H}$.
What about real quadratic fields? Not so easy! *Idea: (Darmon–Vonk, 2021)*

- ∞ is inert in CM-fields, but splits in RM-fields. So work at an inert place $p < \infty$ in an RM-field.
- Change group from $SL_2(\mathbb{Z})$ to $SL_2(\mathbb{Z}[1/p])$ and consider the non-zero meromorphic functions $\mathcal{M}_p^\times$ on $\mathcal{H}_p = \mathbb{P}^1(\mathbb{C}_p) \setminus \mathbb{P}^1(\mathbb{Q}_p)$.
- But $H^0(SL_2(\mathbb{Z}[1/p]), \mathcal{M}_p^\times)$ contains only constants...
- They find cocycles in $H^1(SL_2(\mathbb{Z}[1/p]), \mathcal{M}_p^\times)$ mirroring singular moduli.

Important: algebraicity statements are still conjectural!

Rigid meromorphic cocycles (2/2)

Darmon–Gehrmann–Lipnowski (2025)

Let $V/\mathbb{Q}$ be any quadratic space of signature (r, s). Then there is:

- an archimedean space X_∞ and a rigid p-adic space X_p;
- a notion of special divisors on X_p, constructed using X_∞;
- a notion of rigid meromorphic cocycles on X_p in $H^s(\Gamma, \mathcal{M}_p^\times)$ of functions with these special divisors;
- a notion of special points on X_p where we can evaluate the cocycles.

Motivation for this project

- In signature $(r, s) = (2, 1)$, obtain Darmon–Vonk cocycles. *Algebraicity of RM-values is still conjectural*, as there is no clear connection to geometry.
- If $s = 0$, then elements in $H^0(\Gamma, \mathcal{M}_p^\times)$ are Γ-invariant functions. For example, all $\Theta(\tau_1, \tau_1'; z)$ are rigid meromorphic 0-cocycles.
- Can we prove the algebraicity of their values without using the geometry from CM-theory? Or maybe we can obtain refined information?

Main thesis result

To state main result, need related quantity

$$\Theta_p(\tau_1,\tau_2) = \prod_{\gamma\in\Gamma} \frac{\tau_2-\gamma\pi\tau_1}{\tau_2-\gamma\pi\tau_1'}\frac{\tau_2'-\gamma\pi\tau_1'}{\tau_2'-\gamma\pi\tau_1}$$

where $\pi \in R \subset B$ is a fixed quaternion of norm p.
CM points τ_i on $\Gamma\setminus\mathcal{H}_p$ of discriminant D_i carry an action of $\mathrm{Pic}(K_i)$.

Theorem (D., 2024)

It holds that
$$\prod_{\substack{c_1\in\mathrm{Pic}(K_1)\\ c_2\in\mathrm{Pic}(K_2)}} \frac{\Theta(c_1\cdot\tau_1, c_2\cdot\tau_2)}{\Theta_p(c_1\cdot\tau_1, c_2\cdot\tau_2)} = \prod_{|x|<\sqrt{D}} \mathcal{F}\left(\frac{D-x^2}{4N}\right)^{\delta(x)},$$
where $\delta(x)\in\{\pm1\}$ is a sign determined by $x \mod 4N$.

Corollary

Suppose that $\mathrm{Pic}(K_1)=\{0\}=\mathrm{Pic}(K_2)$. Then $\Theta(\tau_1,\tau_2)/\Theta_p(\tau_1,\tau_2)\in\mathbb{Q}$.

Remark: independently, it should hold that $\Theta(\tau_1,\tau_2)$, $\Theta_p(\tau_1,\tau_2)\in F$.

Idea of the proof

As p is inert in K_1 and K_2, it splits in $F = \mathbb{Q}(\sqrt{D_1 D_2})$. We consider a p-stabilisation of the Hilbert Eisenstein series $E_{\mathbb{1},\chi}$:

$$E_{\mathbb{1},\chi}^{(p)} = (1 - |_{\mathfrak{p}_1})(1 + |_{\mathfrak{p}_2})\, E_{\mathbb{1},\chi}.$$

This choice of signs ensures that $E_{\mathbb{1},\chi}^{(p)}$ is a p-adic *cusp form*.

We look for a *cuspidal family* of Hilbert modular forms through $E_{\mathbb{1},\chi}^{(p)}$; then

$$e^{\mathrm{ord}} \left(\frac{d}{ds} E_{\mathbb{1},\chi}^{(p)}(z, z, s)\bigg|_{s=0} \right) \in S_2(\Gamma_0(N)) \approx 0.$$

But writing down explicit cuspidal families is hard. Idea:

- Consider its associated Galois representation $\mathbb{1} \oplus \chi$;
- Deform it infinitesimally ($\varepsilon^2 = 0$) and explicitly;
- *Argue why these deformations are modular*;
- Compute the Fourier coefficients of the diagonal restriction;
- The ε-part then yields a meaningful derivative.

This mimics work by Darmon–Pozzi–Vonk and the proof by Zagier.

Deforming $\mathbb{1} \oplus \chi$

Again let $\rho = \mathbb{1} \oplus \chi$ and $\tilde{\rho}$ be a deformation of ρ to $GL_2(\mathbb{Q}_p[\epsilon])$ where $\epsilon^2 = 0$.

Proposition

Let $a, b, c, d : G_F \to \mathbb{Q}_p$ be those functions such that

$$\tilde{\rho}(\tau) = \left(1 + \epsilon \begin{pmatrix} a(\tau) & b(\tau) \\ c(\tau) & d(\tau) \end{pmatrix}\right) \cdot \rho(\tau)$$

for all $\tau \in G_F$. Then these functions must respectively satisfy

$$a, d \in \mathrm{Hom}(G_F, \mathbb{Q}_p), \quad \text{and} \quad b, c \in H^1(G_F, \mathbb{Q}_p(\chi)).$$

Note that $\dim \mathrm{Hom}(G_F, \mathbb{Q}_p) = 1$ spanned by the p-adic cyclotomic character:

$$\phi_p^{cyc} : G_F \to \mathrm{Gal}(F(\zeta_{p^\infty})/F) \cong \mathbb{Z}_p^\times \xrightarrow{\log_p} \mathbb{Q}_p.$$

We choose

$$\widetilde{\rho} = \left(1 + \epsilon \begin{pmatrix} \phi_p^{cyc} & * \\ 0 & -\phi_p^{cyc} \end{pmatrix}\right) \begin{pmatrix} 1 & * \\ 0 & \chi \end{pmatrix}.$$

The approach in a nutshell

$$
\begin{tikzcd}
\text{Eisenstein series } E^{(p)}_{\mathbb{1},\chi} \arrow[r, "\text{classical}"] \arrow[d, dashed] & \rho = \begin{pmatrix} 1 & 0 \\ 0 & \chi \end{pmatrix} \arrow[d] \\
\text{Family } E^{(p)}_{\mathbb{1},\chi}(\epsilon) = E^{(p)}_{\mathbb{1},\chi} + \epsilon E' & \widetilde{\rho} = \left(1 + \epsilon \begin{pmatrix} \phi_p^{\mathrm{cyc}} & * \\ 0 & -\phi_p^{\mathrm{cyc}} \end{pmatrix}\right) \begin{pmatrix} 1 & * \\ 0 & \chi \end{pmatrix} \arrow[l, "R = T"']
\end{tikzcd}
$$

The Fourier coefficients of $E^{(p)}_{1,\chi}(\epsilon)$ are obtained from Frobenius traces at $\ell \neq p$:

$$\mathrm{tr}(\tilde{\rho}(\mathrm{Frob}_{\mathfrak{l}})) = \begin{cases} 2 & \text{if } \chi(\mathfrak{l}) = 1; \\ 2\log_p(\mathrm{Nm}(\mathfrak{l}))\epsilon & \text{if } \chi(\mathfrak{l}) = -1. \end{cases}$$

- At p, use the quotient characters of the *nearly ordinary* deformation.
- The deformation is in the *anti-parallel weight direction*.
- Consider diagonal restriction with respect to the ideal class of $\mathcal{D}_F^{-1}\mathfrak{q}$.

Technical step: need to justify why $\widetilde{\rho}$ comes from a family of cusp forms, which requires proving an $R = T$-theorem. *(Betina, Dimitrov, Shih, Deo...)*

Rewriting the Θ-functions

Both the numerator and denominator live inside F:

$$\Theta(\tau_1, \tau_2) = \prod_{\gamma \in \Gamma} \frac{\tau_2 - \gamma\tau_1}{\tau_2 - \gamma\tau_1'} \frac{\tau_2' - \gamma\tau_1'}{\tau_2' - \gamma\tau_1} = \lim_{n \to \infty} \prod_{\substack{\nu \in F^+ \\ \mathrm{tr}(\nu) = p^{2n}}} \left(\frac{\nu}{\nu'}\right)^{n_\nu},$$

where ν' is the Galois conjugate of ν, and where all $n_\nu \geqslant 0$.

Theorem (D., 2026)

Let $\mathfrak{q}$ be a prime of F above $q \mid N$. Then for some $C \in \mathrm{Pic}(L)$,

$$n_\nu = \#\left\{ I \subset \mathcal{O}_L \mid \mathrm{Nm}_F^L(I) = (\nu)\mathcal{D}_F\mathfrak{q}^{-1} \text{ and } [I] = C \right\}.$$

There is an exact sequence

$$\mathrm{Pic}(K_1) \times \mathrm{Pic}(K_2) \to \mathrm{Pic}(L) \xrightarrow{\mathrm{Nm}_F^L} \mathrm{Pic}(F)^+ \xrightarrow{\chi} \{\pm 1\} \to 1.$$

Taking the average over all CM points, the exponent becomes

$$\#\left\{ I \subset \mathcal{O}_L \mid \mathrm{Nm}_F^L(I) = (\nu)\mathcal{D}_F\mathfrak{q}^{-1} \right\} = \text{Fourier coefficient of } E_{\mathbb{1},\chi}.$$

Class number 2 examples

Let $\mathrm{Pic}(K_1) = \{0\}$ and $\mathrm{Pic}(K_2) = \mathbb{Z}/2\mathbb{Z}$ and take $N = 6$.

Giampietro–Darmon's prediction

The values $\Theta(\tau_1, \tau_2) \in H_1H_2$ are of degree $2h_1h_2 = 4$ over $\mathbb{Q}$.

- If $(D_1, D_2) = (-19, -91)$, then $\Theta(\tau_1, \tau_2)$ satisfies the polynomial
$$2690048x^2 - 14175811x + 5797376 \implies F = \mathbb{Q}(\sqrt{19 \cdot 91}).$$
- If $(D_1, D_2) = (-43, -123)$, then $\Theta(\tau_1, \tau_2)$ satisfies the polynomial
$$\begin{aligned} &281165824x^4 - 338910935382x^3 + 1070743516317x^2 \\ &- 239370828534x + 3401222400 \implies \mathbb{Q}(\sqrt{41}, \sqrt{129}) \supset F. \end{aligned}$$
- If $(D_1, D_2) = (-19, -187)$, then $\Theta(\tau_1, \tau_2)$ satisfies the polynomial
$$\begin{aligned} &11017768638322092015616x^4 - 79591235287463638422528x^3 \\ &+ 208185717916774123303311 3x^2 - 62343682036111083304652 8x \\ &+ 760770736060943994716 16 \implies \mathbb{Q}(\sqrt{-11}, \sqrt{-323}) \supset F. \end{aligned}$$

Understanding the degree

Recall from the rewriting step that for a class $C \in \mathrm{Pic}(L)$ with $\mathrm{Nm}^L_F(C) = [\mathcal{D}_F \mathfrak{q}^{-1}] \in \mathrm{Pic}(F)^+$, we may define

$$\Theta_C := \lim_{n\to\infty} \prod_{\substack{\nu \in F^+ \\ \mathrm{tr}(\nu) = p^{2n}}} \left(\frac{\nu}{\nu'}\right)^{\rho_C\left((\nu)\mathcal{D}_F \mathfrak{q}^{-1}\right)},$$

where $\rho_C(J) = \#\{I \subset \mathcal{O}_L \mid \mathrm{Nm}^L_F(I) = J \text{ and } [I] = C\}$. If $\mathrm{Gal}(L/F) = \langle \sigma_F \rangle$, then

$$\rho_C(J) = \rho_{\sigma_F(C)}(J) \implies \Theta_C = \Theta_{\sigma_F(C)}.$$

Class number 2 case

Suppose that $\mathrm{Pic}(K_1) = \{0\}$ and $\mathrm{Pic}(K_2) = \mathbb{Z}/2\mathbb{Z}$ and let $C_1 \neq C_2 \in \mathrm{Pic}(L)$ be the two classes whose norm to F is $[\mathcal{D}_F \mathfrak{q}^{-1}]$. If $\sigma_F(C_1) = C_2$, we expect

$$\Theta_{C_1} = \Theta_{C_2} \in F.$$

More refined deformations

Why do deformations of $\mathbb{1} \oplus \chi$ compute the norm? Note that

$$\mathbb{1} \oplus \chi = \mathrm{Ind}_L^F(\mathbb{1}) \implies \sum_{c_1,c_2} \log_p\left(\frac{\Theta(c_1 \cdot \tau_1, c_2 \cdot \tau_2)}{\Theta_p(c_1 \cdot \tau_1, c_2 \cdot \tau_2)}\right) = \log_p\left(\text{“Nm”}(\Theta(\tau_1,\tau_2))\right).$$

Key idea

Consider $\rho = \mathrm{Ind}_L^F(\psi)$, where $\psi : \mathrm{Pic}(L) \to \mathbb{C}_p^\times$ is a character. Then ρ corresponds to a Hilbert modular θ-series, and deformations of ρ should yield

$$\sum_{c_1,c_2} \psi(c_1 c_2) \log_p\left(\Theta(c_1 \cdot \tau_1, c_2 \cdot \tau_2) \cdot \Theta_p(c_1 \cdot \tau_1, c_2 \cdot \tau_2)^{\pm 1}\right).$$

Values for all ψ yields (using orthogonality) information about one $\Theta(\tau_1,\tau_2)$.

The Hilbert modular θ-series has four p-stabilisations,

$$\theta_{++}, \theta_{--} \quad \text{and} \quad \theta_{-+}, \theta_{+-},$$

which correspond to $\Theta(\tau_1,\tau_2) \cdot \Theta_p(\tau_1,\tau_2)$ and $\Theta(\tau_1,\tau_2)/\Theta_p(\tau_1,\tau_2)$ resp.

Class number 2 example (1/4)

Let $D_1 = -19$ and $D_2 = -187$. Take $N = 6$ so work on X_6.
Then $\mathrm{Pic}(K_1) = \{0\}$ and $\mathrm{Pic}(K_2) \cong \mathbb{Z}/2\mathbb{Z}$.

n	$(D - x^2)/4N$	Special prime(s)	$\delta(n)$
1	$2^2 \cdot 37$	37	1
5	$3 \cdot 7^2$	3	-1
7	$2 \cdot 73$	2	-1
11	$11 \cdot 13$	13	1
13	$3 \cdot 47$	3	1
17	$2^3 \cdot 17$	2	-1
19	$7 \cdot 19$	19	-1
23	$2 \cdot 3^2 \cdot 7$	2	1
25	$2 \cdot 61$	2	1
29	113	113	-1
31	$2^2 \cdot 3^3$	3	-1
35	97	97	1
37	$7 \cdot 13$	13	1
41	$2 \cdot 3 \cdot 13$	$2, 3, 13$	-1
43	71	71	-1
47	$2^3 \cdot 7$	2	1
49	$2^4 \cdot 3$	3	1
53	31	31	-1
55	$2 \cdot 11$	2	-1
59	3	3	1

Class number 2 example (2/4)

Let $D_1 = -19$ and $D_2 = -187$. Then $\mathrm{Pic}(K_1) = \{0\}$ and $\mathrm{Pic}(K_2) \cong \mathbb{Z}/2\mathbb{Z}$. There are now four values Θ_1, $\Theta_{1,p}$, Θ_2, $\Theta_{2,p} \in \mathbb{Q}(\sqrt{-11}, \sqrt{-323}) \supset F$. In my thesis, I showed

$$\frac{\Theta_1 \cdot \Theta_2}{\Theta_{1,p} \cdot \Theta_{2,p}} = \frac{13^4 \cdot 37 \cdot 97}{3 \cdot 19^2 \cdot 31 \cdot 71 \cdot 113} = \frac{102505429}{269356179}.$$

How to compute norms to $\mathbb{Q}(\sqrt{-11})$ and $\mathbb{Q}(\sqrt{-323})$?

$$\begin{tikzcd}[column sep=small]
 & H_2 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathbb{Q}(\sqrt{-11}) \arrow[dr, dash] & K_2 \arrow[d, dash] & \mathbb{Q}(\sqrt{17}) \arrow[dl, dash] \\
 & \mathbb{Q} &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & \mathbb{Q}(\Theta) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathbb{Q}(\sqrt{-11}) \arrow[dr, dash] & F \arrow[d, dash] & \mathbb{Q}(\sqrt{-323}) \arrow[dl, dash] \\
 & \mathbb{Q} &
\end{tikzcd}$$

Take $p = 3$ and $q = 2$, so $N = 6$. These primes both split in F and:

- the primes $\mathfrak{p}, \mathfrak{p}'$ are *not* principal in $F \implies \Theta$ not totally real;
- the primes $\mathfrak{q}, \mathfrak{q}'$ *are* principal in $F \implies \Theta_1$ degree 4.

Class number 2 example (3/4)

Proposition

Let $a, b, c, d : G_F \to \mathbb{Q}_p$ be those functions such that

$$\tilde{\rho} = \left(1 + \epsilon \begin{pmatrix} a(\tau) & b(\tau) \\ c(\tau) & d(\tau) \end{pmatrix}\right) \cdot \operatorname{Ind}_L^F(\psi) \text{ where } \operatorname{Ind}_L^F(\psi)|_{G_L} = \begin{pmatrix} \psi(\tau) & 0 \\ 0 & \psi^\sigma(\tau) \end{pmatrix}$$

for all $\tau \in G_F$ and where $\sigma = \sigma_F$. Then still $a, d \in \operatorname{Hom}(G_L, \mathbb{Q}_p)$, but now

$$b \in H^1(G_L, \mathbb{Q}_p(\psi_\heartsuit)) \quad \text{and} \quad c \in H^1(G_L, \mathbb{Q}_p(\psi_\heartsuit^{-1})), \quad \text{where} \quad \psi_\heartsuit = \psi/\psi^\sigma.$$

Choose ψ quadratic so that $\psi_\heartsuit$ is also quadratic. Then $\operatorname{Ind}_L^F(\psi)$ defines a theta series θ, and choose an appropriate deformation of θ_{++}. For $\alpha = \sqrt{-11}$,

$$3 = \frac{1-\alpha}{2} \cdot \frac{1+\alpha}{2}, \quad 31 = \frac{5+3\alpha}{2} \cdot \frac{-5+3\alpha}{2}, \quad 37 = \frac{7-3\alpha}{2} \cdot \frac{7+3\alpha}{2},$$
$$71 = \frac{3-5\alpha}{2} \cdot \frac{3+5\alpha}{2}, \quad 97 = \frac{17+3\alpha}{2} \cdot \frac{17-3\alpha}{2}, \quad 113 = \frac{21+\alpha}{2} \cdot \frac{21-\alpha}{2}.$$

We can show that

$$\log_p\left(\frac{\Theta_1 \cdot \Theta_{1,p}}{\Theta_2 \cdot \Theta_{2,p}}\right) = \log_p\left(\frac{1-\alpha}{1+\alpha} \cdot \frac{5+3\alpha}{5-3\alpha} \cdot \frac{7-3\alpha}{7+3\alpha} \cdot \frac{3-5\alpha}{3+5\alpha} \cdot \frac{17+3\alpha}{17-3\alpha} \cdot \frac{21-\alpha}{21+\alpha}\right).$$

Class number 2 example (4/4)

Computing the norm to $\mathbb{Q}(\beta)$, where $\beta = \sqrt{-323}$, uses θ_{+-}:

$$\log_p\left(\frac{\Theta_1 \cdot \Theta_{2,p}}{\Theta_{1,p} \cdot \Theta_2}\right) = \frac{1}{2}\log_p\Big(\frac{1+\beta}{18} \cdot \frac{1903+15\beta}{1922} \cdot \frac{-1063+48\beta}{1369} \cdot \frac{2609+240\beta}{5041} \cdot \frac{15607+585\beta}{18818} \cdot \frac{-18679+969\beta}{25538}\Big).$$

Recall that we can compute

$$\frac{\Theta_1 \cdot \Theta_2}{\Theta_{1,p} \cdot \Theta_{2,p}} = \frac{13^4 \cdot 37 \cdot 97}{3 \cdot 19^2 \cdot 31 \cdot 71 \cdot 113} = \frac{102505429}{269356179}$$

and

$$\log_p\left(\frac{\Theta_1 \cdot \Theta_{1,p}}{\Theta_2 \cdot \Theta_{2,p}}\right) = \log_p\left(\frac{1-\alpha}{1+\alpha} \cdot \frac{5+3\alpha}{5-3\alpha} \cdot \frac{7-3\alpha}{7+3\alpha} \cdot \frac{3-5\alpha}{3+5\alpha} \cdot \frac{17+3\alpha}{17-3\alpha} \cdot \frac{21-\alpha}{21+\alpha}\right).$$

Theorem: D., Yingkun Li (2026)

Suppose T is an imaginary quadratic subfield of $\mathbb{Q}(\Theta(\tau_1, \tau_2))$. Then we can compute a Gross-Zagier style factorisation formula for $\mathrm{Nm}_T(\Theta(\tau_1, \tau_2)) \in T$.

Currently working on the general case beyond quadratic characters.

Thank you for your attention!

n	$(D - x^2)/4N$	Special prime(s)	$\delta(n)$
1	$2^2 \cdot 37$	37	1
5	$3 \cdot 7^2$	3	-1
7	$2 \cdot 73$	2	-1
11	$11 \cdot 13$	13	1
13	$3 \cdot 47$	3	1
17	$2^3 \cdot 17$	2	-1
19	$7 \cdot 19$	19	-1
23	$2 \cdot 3^2 \cdot 7$	2	1
25	$2 \cdot 61$	2	1
29	113	113	-1
31	$2^2 \cdot 3^3$	3	-1
35	97	97	1
37	$7 \cdot 13$	13	1
41	$2 \cdot 3 \cdot 13$	$2, 3, 13$	-1
43	71	71	-1
47	$2^3 \cdot 7$	2	1
49	$2^4 \cdot 3$	3	1
53	31	31	-1
55	$2 \cdot 11$	2	-1
59	3	3	1